Roshdi Rashed (ed.)
Oeuvre mathématique d'al-Sijzī.
Volume 1 : *Géométrie des coniques et théorie des nombres au x^e siècle*

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Roshdi Rashed, Pascal Crozet (eds.)
Oeuvre mathématique d'al-Sijzī. Volume II : *Géométrie et Philosophie des Mathématiques au x^e siècle*

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Aḥmad ibn Muḥammad ibn ʿAbd al-Jalīl al-Sijzī was a geometer, astronomer and astrologer who worked in the second half of the tenth century AD in Iran and surrounding areas. Although he did not belong to the most creative mathematicians in the Islamic tradition, he was competent, and a large part of his geometrical oeuvre has survived. The books under review contain editions of the original Arabic texts and French translations of roughly two-thirds of his extant work in geometry. Most of the texts had not been edited and translated before. Taken together this material provides new insights in the geometrical tradition in Iran in the second half of the tenth century AD, when the study of geometry was at its peak. The two editors of the volumes under review are specialists in the field. Roshdi Rashed, retired historian of science, is well-known among historians of mathematics for the astonishing number of critical editions and French translations of Arabic mathematical texts which he has published. Pascal Crozet is a specialist in the history of geometry between the ninth and eleventh centuries, who has worked on al-Sijzī for many years.

In the introduction to volume 2, the editors have collected and analyzed the available information about al-Sijzī's life. Al-Sijzī was born in Sijistān, also called Sīstān, an area in the South-East of Iran and the adjacent part of Afghanistan. His father Muḥammad ibn ʿAbd al-Jalīl al-Sijzī was also a geometer, and thus the mathematical activities of the son Aḥmad seem to have started at a young age. He was active in a period spanning from 960 AD to after 1000 AD, and the editors have identified the titles of 47 works by al-Sijzī on geometry, of which no less than 37 are extant today. Here "work" can mean anything between a letter of a few pages and a substantial treatise in 50 pages or more. By means of al-Sijzī's references to his own works and the dates which he added to some of them, the editors have produced a tentative chronological division of the 37 extant works and some of the lost works as well. Al-Sijzī had a large scientific network and he seems to have had students: the *Introduction to Geometry* in vol. 2 is specifically written for "beginners" in geometry and also in the *Treatise on the Heptagon* (vol. 1, p. 401) al-Sijzī says that he wants to explain the situation with all its controversies to the "beginners".

A large number of his extant works have come down in manuscripts that were copied and studied in the Niẓāmiyya madrasa in Baghdad in the twelfth and thirteenth century AD. One such manuscript, copied in 1159-1162, has been preserved in a private library in Lahore, and another manuscript, copied in 1215 AD, is partly extant in the Chester Beatty Library in Dublin (ms. Arabic 3652). According to the index (published in vol. 2, p. 856-861), this manuscript codex once contained 38 works by al-Sijzī. In addition, the Bibliothèque Nationale in Paris possesses a manuscript (Fonds Arabe 2457) which is believed to be in al-Sijzī's own handwriting. The quality of the geometrical figures in this Paris manuscript is not very good, so if this manuscript was indeed written by al-Sijzī himself, his abilities as a draughtsman were limited.

In the rest of this review, I will go through the treatises by al-Sijzī which are edited and translated in the two volumes and will briefly explain some interesting features of these works. The edited Arabic texts and French translations appear on facing pages in the two volumes and were done with care.

Volume 1 begins with introductions and commentaries. The volume deals with al-Sijzī's writings on the three conic sections: the ellipse, parabola and hyperbola. These are the curves that one may see by shining a flashlight on a wall. The names of the conic sections were introduced by Apollonius of Perga (ca. 200 BC), whose major work *Conics* was translated into Arabic and well-known to al-Sijzī. The first two texts in vol. 1 contain new research by al-Sijzī which he addressed to his father. In an earlier treatise, which is now lost, al-Sijzī had studied what he called the "egg" and the "lentil", that is to say the solids which one obtains by rotating an ellipse around its major and minor axis. He asked the question what sort of curves one gets by intersecting the egg and lentil by a plane, and he showed that the intersections can only be circles and ellipses. In the first treatise (vol. 1, p. 189-210) al-Sijzī answers the same question for the parabolic and hyperbolic cupola, obtained by rotating a parabola or hyperbola around its axis. Rashed points out that al-Sijzī's treatment is imperfect: for example, al-Sijzī does not realize that a parabola can also be obtained as intersection of a plane with a hyperbolic cupola. The second treatise (vol. 1, p. 211-228) deals in a similar way with cylindrical solids which can be obtained from the three conic sections.

The next two treatises deal with al-Sijzī's methods for actually drawing conic sections in a plane. The third treatise (vol. 1, p. 229-280) had been briefly summarised by Franz Woepcke in 1851 from a manuscript in Leiden, Netherlands, but had never been published in full, and the fourth treatise (vol. 1, p. 281-292) was recently discovered in a library in India. Drawing conic sections was important for practical purposes as well. Al-Sijzī presents the construction of a horizontal sundial as an example, with a figure (reproduced on vol. 1, p. 274-275) and explicit computations. This is exciting material.

The fifth treatise (vol. 1, p. 293-310) is expository, on the asymptotes of the hyperbola, and is inspired by Book 2 of Apollonius' *Conics*. The sixth treatise (vol. 1, p. 311-332) is a letter to Naṣr ibn ʿAbdallāh otherwise unknown, on the notion that all geometrical figures are derived from the circle. The contents of the letter are philosophical rather than mathematical and in this sense the treatise is different from the preceding ones, although al-Sijzī uses geometrical theorems in his argument. Next comes (vol. 1, p. 333-386) a long treatise on the ancient Greek problem of the trisection, that is the division of a given angle into three equal parts. The treatise had also been summarized by Franz Woepcke in 1851 but it had never been published in full. The trisection of the angle cannot in general be executed by ruler and compass. Al-Sijzī presents preliminary discussions of this problem by five Arabic-Islamic mathematicians, all mentioned by name, and he presents two solutions by means of a circle and a hyperbola. One of the solutions is ultimately of ancient Greek origin while the other solution was found by Abū Sahl al-Kūhī in the late tenth century. This is followed by a short treatise (vol. 1, p. 387-396) with one trisection of the angle and one construction of two mean proportionals, both by means of a hyperbola, and ultimately of ancient Greek origin.

The subsequent treatise (vol. 1, p. 397-420) is a previously published work on the construction of the regular heptagon, also a problem that can not be solved by the Euclidean ruler and compass. Al-Sijzī presents a construction of this figure by means of conic sections which he adapted from his contemporary Al-ʿAlā ibn Sahl. The introduction contains scathing criticisms of a mathematician whose name is not mentioned but who turns out to be Abū'l-Jūd Muḥammad ibn al-Layth. The criticism was based on the fact that Abū'l-Jūd had presented an incorrect construction of the regular heptagon by means of ruler and compass. Al-Sijzī also complains that the unnamed geometer (Abū'l-Jūd) had criticized a construction of the heptagon by Archimedes, even though the Archimedean construction was not acceptable according to al-Sijzī's own criteria either. Of course there may have been personal issues involved.

The rest of volume 1 is devoted to what the editor calls "number theory". One more hitherto unpublished treatise (vol. 1, p. 421-454) concerns the construction of right-angled triangles with rational sides. This leads to the problem to construct two square numbers whose sum is also a square. The problem had been completely solved in antiquity and al-Sijzī's treatise is expository. Two modest fragments on number problems and indices conclude volume 1.

The first text in the second volume (vol. 2, p. 52-197) is al-Sijzī's *Introduction to Geometry*. This previously unpublished text is probably the most interesting work by al-Sijzī for general historians and philosophers of science and will undoubtedly inspire research in the history of Arabic-Islamic science. The *Introduction to Geometry* is addressed to beginners, who are studying the *Elements* of Euclid but do not have a clear idea what geometry is about. Al-Sijzī wants to present them with a perspective on geometry as a whole, so that they are motivated to continue their studies. Al-Sijzī first explains basic concepts of geometry and he then continues by presenting geometrical theorems with figure and without proof. The theorems were obviously selected on the basis of their intrinsic interest or beauty, and thus al-Sijzī makes clear why he loves geometry so much. He also mentions the fields in which geometry can be applied, although he indicates that the first and foremost use of geometry is the training of the human mind (vol. 2, p. 54-55). The work shows the deep influence of Greek geometry in 10th-century Iran, and at the same time that the subject was even more restricted than in Greek

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antiquity. The only curves that al-Sijzī allows are the circle, the three conic sections (parabola, hyperbola and ellipse), and as a boundary case the cylindrical helix which has order and regularity but cannot be measured (vol. 2, p. 58-59). All other curves do not belong to geometry since they cannot be measured and have no regularity (vol. 2, p. 60-61). Thus many ancient Greek curves such as the conchoid, the quadratrix, the cissoid, etc. are excluded. The only three-dimensional solids with plane surfaces which al-Sijzī presents are the five regular polyhedra, the prism, the parallelepiped, and the general tetrahedron. There is no concept of an arbitrary polyhedron; nor are the semi-regular Archimedean polyhedra mentioned. Solids with curved surfaces are also severely restricted, and as Rashed and Crozet remark (vol. 2, p. 98, footnote 27), al-Sijzī does not even consider the oblique cone – even though it plays a crucial role in the *Conics* of Apollonius, a work which he cites and uses all the time. Al-Sijzī bluntly states that "the surface of the torus and its solid are not used in this art and therefore we do not mention them" (vol. 2, p. 96-97), even though the torus is used in an ancient Greek construction of two mean proportionals which had been transmitted into Arabic. It is important to understand the role of such conscious or unconscious restrictions in Arabic-Islamic geometry, also in the work of later authors. For example, in a treatise on isoperimetry, Ibn al-Haytham (ca. 965-1041) wants to show that the sphere is the largest solid with a given (constant) surface area. Because he compares the sphere with only a few other solids, Ibn al-Haytham's argument may seem woefully inadequate from a modern point of view, until we understand the role of the restrictions of Arabic-Islamic geometry, in which the concept of an arbitrary polyhedron with plane surfaces had not been developed.

The most important application of geometry was of course astronomy, and because al-Sijzī was an astrologer, it is interesting to see what he has to say on astrology (vol. 2, p. 56-57). He says that some people consider astrology as a higher science than geometry, and he continues that astrology is not an exact science, and that its laws are weak and unconvincing. He then complains that many people who do not even know the letters of the alphabet concern themselves with astrology. He ends the discussion with some ancient Greek quotations including the Platonic dictum "let no one ignorant of geometry enter here".

The applications which al-Sijzī discusses include architecture, but he does not mention the decorative geometrical patterns which are often associated with Islamic culture, as seen on the cover of this journal. Al-Sijzī only has a cryptic sentence that "all instruments used by craftsmen are based on geometry, even though they do not know it. I once saw a man from 'Askar, Sijistan, who said that he had seen a book on the foundations of crafts and the existence of a geometry for all of them in an artisanal way and mentioned some of it" (vol. 2, p. 194-195). For al-Sijzī, geometry could only be taken seriously if it was based on proofs as in the *Elements* of Euclid. There may have been practical methods for constructing decorative patterns in 10th century Iran that were despised by scholars such as al-Sijzī because they were approximate only and therefore could not be proved.

It would be possible to say much more on the *Introduction to Geometry*, but I hope that the reader will now have an impression of the richness of the work.

In the second treatise (vol. 2, p. 200-213), al-Sijzī analyses and extends ancient Greek proofs concerning the irrationality of the ratio between the side and diagonal of a square. Surprisingly, he does not present the ancient Greek numerical approximations of the ratio by means of "side and diagonal numbers" (1/1, 2/3, 5/7, 12/17, etc.).

In the next part (vol. 2, p. 225-457), the editors have collected al-Sijzī's discussions of propositions in the different books of Euclid's *Elements*. These extensive and detailed discussions will interest all students of the history of Euclid's *Elements*. The material also shows different interactions with other mathematicians. For Naẓīf ibn Yumn, al-Sijzī compiled thirty different proofs of proposition 10 in Book 4 of the *Elements*, related to the golden section and the decagon (vol. 2, p. 304-343). Elsewhere al-Sijzī criticized a geometrical proof by Yuḥannā ibn Yūsuf at the request of a local king (vol. 2, p. 450-457), showing that Euclid's *Elements* were also studied in royal circles.

The longest text in the two volumes (vol. 2, p. 466-737, 842-853) is a collection of problems entitled *The selected problems that were discussed between him* (al-Sijzī) *and the geometers of Shīrāz and Khorāsān, and his annotations*. The title was obviously modelled on the *Selected Problems* by Ibrahīm ibn Sinān (909-946 AD), which work is cited by al-Sijzī (vol. 2, p. 730-731), with the difference that al-Sijzī's selected problems were more in number and usually less complicated. The two editors have produced the first complete edition and translation of this highly interesting text. Some of al-Sijzī's selected problems had been published before by other researchers in articles which are mentioned only in the bibliography of vol. 2, not in the edition and translation itself. Al-Sijzī presents

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115 geometrical problems with mostly brief solutions by ruler and compass geometry. As the editors point out in the introduction, the problems appear without any specific order. The collection must have been compiled around 972 AD, rather early in al-Sijzī's life, when his father was still alive. The treatise is interesting not only because of al-Sijzī's own solutions, but also because of the information which it contains about al-Sijzī's knowledge of the literature, the discussions in geometry in Iran at the time, and the contacts between al-Sijzī and other mathematicians. It turns out that al-Sijzī possessed Arabic versions of several minor works of Apollonius of Perga which are now lost, and which were known to European mathematicians in the 17th century only through the incomplete descriptions by Pappus of Alexandria (ca. 300 AD). Thus, European mathematicians sought to "reconstruct" these works by Apollonius which al-Sijzī still possessed in the original. One probable example is the Apollonian treatise on tangent lines and circles, with the solution by Apollonius of the famous problem to construct a circle tangent to three given circles. Al-Sijzī even wrote a treatise, now lost, on the "demonstration of the propositions in the work of Apollonius on circles" (vol. 2, p. 860).

The *Selected Problems* are followed by two related and previously unpublished texts, *On obtaining certain geometrical laws* (vol. 2, p. 739-762), and *On the properties of the diameter of a circle* (vol. 2, p. 763-775).

The last text is a mysterious treatise (vol. 2, p. 800-839) entitled *Answers to ten problems which were asked to him by some geometer from Shīrāz*, and probably addressed by al-Sijzī to one of his students. The problems concern the division of figures (parallelograms, etc.) into parts according to a well-defined pattern where the surfaces are in a certain ratio. The editors point out that these problems belong to a tradition about which not much is known in the modern historical literature. The eighth problem, on the division of a parallelogram into four parts, could somehow be related to practical problems in the construction of decorative patterns, but such patterns are nowhere mentioned. Al-Sijzī says that the ninth problem is ill-posed because there are infinitely many solutions, and he makes the curious statement that "it is impossible to solve indeterminate problems" (vol. 2, p. 836-837).

Volume 2 closes with indices of technical terms and a bibliography. We finish this review by making a few relevant bibliographical additions to the volumes. With reference to volume 1, p. 315-331, the same text was edited in Arabic as nº. 8 (17 p.) in *Rasāi'lu'l-mutafarriqa f'il-hai'at li-l mutaqaddimīn wa mu'āsiray il-Bīrūnī, containing Eleven Important Treatises on Astronomy and other subjects, Contributed by the famous Predecessors and Contemporaries of al-Biruni*, edited and published by the Dáira tu'l Maʿarifi'l-Osmania, Hyderabad, Deccan, 1948. To the bibliography in vol. 2, add J.L. Berggren, J.P. Hogendijk, "The Fragments of Abū Sahl al-Kūhī's Lost Geometrical Works in the Writings of al-Sijzī", p. 605-665 in: C. Burnett, J.P. Hogendijk, K. Plofker, M. Yano, eds., *Studies in the History of the Exact Sciences in Honour of David Pingree*, Leiden, Brill, 2003, Islamic Philosophy, Theology and Science, vol. 54. This includes Arabic editions and English translations of all problems and their solutions by al-Kūhī in the treatise *On the selected problems* in vol. 2, for example the six problems and solutions in p. 642-655.

In vol. 2 the editors present their list of 47 works by al-Sijzī and they announce that they will publish some remaining works by al-Sijzī (nos. 9, 13, 31, 33-35, 39, 41-44) in a future volume. For the reader who wants to have an idea of these texts we refer to the bibliographical references in the list, to the bibliography in the two volumes, and also to the following publications, not mentioned there: a substantial part of text nº. 9 was edited in Arabic and translated into English in Emre Coşkun, "Thābit ibn Qurra's translation of the *Maʾkhūdhāt Mansūba ilā Arshimīdis*", *SCIAMVS* 19 (2018), p. 53-102. The complete text nº. 31 was edited in Arabic and translated into English in Jan P. Hogendijk, "Traces of the lost Geometrical Elements of Menelaus in two texts of al-Sijzī", *Zeitschrift für Geschichte der arabischislamischen Wissenschaften* 13 (2000), p. 129-164. The complete text nº. 33 was edited in Arabic in the volume *Rasāi'lu'l-mutafarriqa f'il-hai'at li-l mutaqaddimīn wa mu'āsiray il-Bīrūnī* mentioned above. Detailed summaries of text nº. 34 can be found in p. 49-53 of Axel Björnbo, *Thābits Werk über den Transversalensatz, ... herausgegeben und ergänzt durch Untersuchungen über die Entwicklung der muslimischen sphärischen Trigonometrie von H. Bürger und K. Kohl*, Erlangen 1924 (also in p. 30-32 of the article by J.L. Berggren which the editors do mention in connection with text nº. 33).

In conclusion, the two volumes present a wealth of interesting new material on the history of geometry in tenth-century Iran and surrounding areas. The two volumes will be a basic reference for researchers in the exact sciences in Islamic civilisation, and they will be fascinating reading for all historians of ancient, medieval and early modern geometry.

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